中山大学学报(自然科学版)
第 39 卷 第 1 期 ACTA SCIENTIARUM NATURALIUM Vol. 39 No. 1
2000 年 1 月 UNIVERSITATIS SUNYATSENI Jan. 2000

Article ID: 0529-6579(2000)01-0023-04

Representation Space and Holomorphic Bundles*

HU Xin-min
(Department of Mathematics, Zhongshan University, Guangzhou 510275)

Abstract: The representation space of the fundamental group of a closed, oriented Riemannian surface is identified with the moduli space of the holomorphic bundles of rank 2 with trivial determinant bundles over the surface and these spaces are proved to carry a projective structure.

Keywords: moduli space; unitary representation; (semi-) stable

CLC number: O189.3; O152.6 **Document code**: A

1 Representation space and the moduli space

Casson used the representation space of the fundamental group of a closed, oriented surface in *SU*(2) *to define the so-called Casson invariant for the homology 3-spheres. Donaldson*[1] *proved: the space of the irreducible representations of the fundamental group of a closed, oriented surface in SU*(2) *can be identified with the moduli space of stable holomorphic vector bundles of rank 2 and with trivial determinant bundles. Atiyah et al discussed the deep connections between this moduli space and the Yang-Mills theory. In this paper, Donaldson's result to semi-stable bundles is extended and it's proved that the moduli space is a projective variety.*

For the definitions of (semi-) stable holomorphic bundles and related notations, please refer to paper [2].

Let Σ *be a closed, oriented Riemann surface and denote by* V_{ss}^0 *the set of semi-stable holomorphic bundles over* Σ *of rank 2 and with trivial determinant bundles and by* M_0 *the set of equivalence classes of* V_{ss}^0 *under S-equivalence denote by* M *the set of the conjugate classes of the representations of* $\pi_1(\Sigma)$ *in SU*(2). *Then the following main theorem is obtained.*

Theorem As a set, $\mathrm{M}_0 = \mathrm{M}$

In order to prove this theorem, the following lemmas are suggested.

Lemma 1 Let $\rho_1, \rho_2 \in \mathrm{Hom}(\pi_1(\Sigma), SU(2))$. *Then the holomorphic SL*(2, $\boldsymbol{C}$)-principal bundles $P(\rho_1)$ *and* $P(\rho_2)$ *corresponding to* ρ_1 *and* ρ_2 *are isomorphic if and only if the representations* ρ_1 *and* ρ_2 *are conjugate.*

Proof If ρ_1 *is conjugate to* ρ_2, *then it is easy to see that* $P(\rho_1)$ *is isomorphic to* $P(\rho_2)$. *On the other hand, if* $P(\rho_1)$ *and* $P(\rho_2)$ *are isomorphic, there exists a holomorphic function*

---

* 基金项目: 中山大学高等学术研究中心基金资助项目(96M5, 028C601)
收稿日期: 1999-05-07 作者简介: 胡新民(1965 ~), 男, 讲师.

$f: \tilde{\Sigma} \to SL(2, \boldsymbol{C})$

such that $\boldsymbol{f}(\boldsymbol{m}\alpha) = \rho_1^{-1}(\alpha)\boldsymbol{f}(\boldsymbol{m})\rho_2(\alpha)$

where $\alpha \in \pi_1(\Sigma)$, $\boldsymbol{m} \in \tilde{\Sigma}$ *and* $\tilde{\Sigma}$ *is the universal covering space of* Σ.

Now consider the positive definite Hermite form $tr(\boldsymbol{A}^*\boldsymbol{A})$ on SL(2, $\boldsymbol{C}$), where $\boldsymbol{A} \in$ SL(2, $\boldsymbol{C}$). Define $\varphi(\boldsymbol{m}\alpha) \stackrel{\text{def}}{=\!=} \|\boldsymbol{f}(\boldsymbol{m}\alpha)\|^2 = \mathbf{tr}(\rho_2(\alpha)^*\boldsymbol{f}(\boldsymbol{m})^*\rho_1^{-1}(\alpha)^*\rho_1^{-1}(\alpha)\boldsymbol{f}(\boldsymbol{m})\rho_2(\alpha)) =$

$$\mathbf{tr}(\boldsymbol{f}(\boldsymbol{m})^*\boldsymbol{f}(\boldsymbol{m})) = \|\boldsymbol{f}(\boldsymbol{m})\|^2 = \varphi(\boldsymbol{m})$$

Hence, φ *is a holomorphic function on* Σ. *Because* Σ *is compact and connected,* φ *is a constant function. Because* $\boldsymbol{f}: \tilde{\Sigma} \to \mathbf{SL}(2, \boldsymbol{C})$ *is holomorphic and* $\|\boldsymbol{f}(\boldsymbol{m})\|^2$ *is constant,* $\boldsymbol{f}$ *must be constant on* $\tilde{\Sigma}$ *by the Maximum Principle. Let this value be* $\boldsymbol{A}$, then

$$\boldsymbol{A} = \rho_1^{-1}(\alpha)\boldsymbol{A}\rho_2(\alpha), \quad \forall \alpha \in \pi_1(\Sigma)$$

It can be shown that $\boldsymbol{A} = \boldsymbol{UB}$, where $\boldsymbol{U} \in U(2)$, and $\boldsymbol{B}$ is a triangular complex matrix with positive real diagonals.

By $\det(\boldsymbol{A}) = 1$, $\det(\boldsymbol{B}) = \lambda > 0$, one has that $\boldsymbol{U} \in SU(2)$. In fact, from the facts that $\boldsymbol{U}^*\boldsymbol{U} = \boldsymbol{I}$ (the identity matrix) and $\det(\boldsymbol{U}) = 1/\lambda$, one has $\lambda = 1$.

It is easy to see that $\rho_1(\alpha) = \boldsymbol{U}^{-1}\rho_2(\alpha)\boldsymbol{U}$, $\forall \alpha \in \pi_1(\Sigma)$

Lemma 2[3] Let $\boldsymbol{V}$, $\boldsymbol{W}$ be two semi-stable holomorphic vector bundles over Σ *with the same rank and degree and one of them be stable. If* $f: \boldsymbol{V} \to \boldsymbol{W}$ *is a nonzero bundle mapping, then* f *is an isomorphism.*

Proof of theorem If $\boldsymbol{V} \in \boldsymbol{V}_{ss}^0$ *is stable, then by paper* [2], $\boldsymbol{V}$ can be obtained from a 2-dimensional irreducible unitary representation ρ of $\pi_1(\Sigma)$, *i.e.*, $\boldsymbol{V} = \tilde{\Sigma} \times_\rho \boldsymbol{C}^2$. *Because* $det(\boldsymbol{V})$ is holomorphically trivial, $\rho \in Hom(\pi_1(\Sigma), SU(2))$.

Let $\boldsymbol{V}_1$, $\boldsymbol{V}_2 \in \boldsymbol{V}_{ss}^0$ *be S-equivalent. By Lemma* 1, *if* $\boldsymbol{V}_1$, $\boldsymbol{V}_2 \in \boldsymbol{V}_{ss}^0$ *are stable and* $gr\mathrm{V}_1 = gr\mathrm{V}_2$, *then their corresponding representations are conjugate.*

If $\boldsymbol{V} \in \boldsymbol{V}_{ss}^0$ *is not stable, there exists a holomorphic line bundle* $\boldsymbol{L} \subset \boldsymbol{V}$ *such that the degrees of* $\boldsymbol{L}$ and $\boldsymbol{V}/\boldsymbol{L}$ are zero. After a suitable choice of an open covering $\{\boldsymbol{U}_i\}$ **of** Σ, *the transition function of* $\boldsymbol{V}$ is of the following form: $\begin{pmatrix} \varphi_{ij} & * \\ 0 & \psi_{ij} \end{pmatrix}$, *where* φ_{ij} (**or** ψ_{ij}) *is the transition functions of* $\boldsymbol{L}$ (or $\boldsymbol{V}/\boldsymbol{L}$).

Since $\det(\boldsymbol{V})$ is holomorphically trivial, so $\varphi_{ij}\psi_{ij} = 1$. *Therefore, if* $\rho: \pi_1(\Sigma) \to \boldsymbol{U}(1) = \boldsymbol{S}^1$ *is the representation corresponding to* L, *then the representation corresponding to* $gr\boldsymbol{V} = \boldsymbol{L} \oplus \boldsymbol{V}/\boldsymbol{L}$ is of the form: $\rho \oplus \rho^{-1}$, *and from Lemma* 1, *if* $gr\boldsymbol{V}_1 = \mathrm{gr}\boldsymbol{V}_2$, then their corresponding representations are conjugate. Now, let ρ_1, $\rho_2 \in \mathbf{Hom}(\pi_1(\Sigma), \boldsymbol{SU}(2))$ *be conjugate.*

1) *If* ρ_1, ρ_2 *are irreducible representations, then their corresponding holomorphic vector bundles are isomorphic.*

2) *If* ρ_1, ρ_2 *are reducible, then* ρ_1, ρ_2 *are conjugate to some elements in Hom* $(\pi_1(\Sigma), \boldsymbol{U}(1))$, *respectively, and it is easy to see that their corresponding holomorphic vector bundles are semi-stable and isomorphic. This completes the proof of theorem.*

2 *A proof of that* M_0 *is a projective variety*

For geometric quotient of an algebraic variety under the actions of an algebraic group, see papers[4, 5].

Proposition　M_0 *is a projective variety*.

Lemma 3[3]　Let $\boldsymbol{E}$ be a semi-stable holomorphic vector bundle of rank n and degree $d > n(2g-1)$ *over* Σ, *where* g *is the genus of* Σ. *Then* $\boldsymbol{E}$ is generated by its global sections, that is, the fiber E_x over $x \in \Sigma$ *is generated by the global sections of* $\boldsymbol{E}$. Moreover one has $H^1(\Sigma, \boldsymbol{E}) = 0$.

By Riemann-Roch Theorem, if $\boldsymbol{E}$ satisfies the conditions of Lemma 3, then $\dim H^0(\Sigma, \boldsymbol{E}) = d + n(1-g)$, *denoted by* p.

Fix a holomorphic line bundle L *of degree* $l > 0$ *over* Σ, *and paper*[2] *gives*

1) $\boldsymbol{E}$ is stable (semi-stable) $\Leftrightarrow \boldsymbol{E} \otimes \boldsymbol{L}$ *is stable* (*semi-stable*);

2) $gr V_1 = gr V_2 \Leftrightarrow gr(V_1 \otimes \boldsymbol{L}) = gr(V_2 \otimes \boldsymbol{L})$

If $V \in V^0_{ss}$, then $\det(V \otimes \boldsymbol{L}) = \det(\boldsymbol{L} \otimes \boldsymbol{L})$, $\det(V \otimes \boldsymbol{L}) = 2l$. Let $M(2, 2l, \boldsymbol{L} \otimes \boldsymbol{L})$ *be the moduli space of the semi-stable holomorphic vector bundles of rank* 2, *degree* 2l *and the determinant bundle* $L \otimes \boldsymbol{L}$ over Σ under S-equivalence. Then $M_0 = M(2, 2l, \boldsymbol{L} \otimes \boldsymbol{L})$. *Now, assume that* $d = 2l$ *satisfies the conditions of Lemma* 3.

For a semi-stable holomorphic vector bundle $\boldsymbol{E}$ with $rk(\boldsymbol{E}) = 2$, $\deg(\boldsymbol{E}) = d > 2(2g-1)$ over Σ, *choose a basis* $S = \{s_1, \cdots, s_p\}$ *of* $H^0(\Sigma, \boldsymbol{E})$, *which will define an isomorphism between* $H^0(\Sigma, \boldsymbol{E})$ and C^p, *denoted by* φ_S.

There exists a natural mapping q: $\wedge^2 H^0(\Sigma, \boldsymbol{E}) \to H^0(\Sigma, \wedge^2 \boldsymbol{E})$, *given by* $(s_i \wedge s_j)(x) = s_i(x) \wedge s_j(x)$.

Now by Lemma 2, *there exists an isomorphism*: $H^0(\Sigma, \wedge^2 \boldsymbol{E}) \to H^0(\Sigma, \boldsymbol{L} \otimes \boldsymbol{L})$. *Moreover, this isomorphism is unique up to a scalar multiplier. Here we assume that* $det(\boldsymbol{E}) = \boldsymbol{L} \otimes \boldsymbol{L}$.

So comes a mapping $T_{E,S}$: $\wedge^2 C^p \to W = H^0(\Sigma, \boldsymbol{L} \otimes \boldsymbol{L})$ *such that the following digram is commutative*:

$$\begin{array}{ccc} \wedge^2 H^0(\Sigma, \boldsymbol{E}) & \xrightarrow{\varphi_S} & \wedge^2 C^p \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle T_{E,S}} \\ H^0(\Sigma, \wedge^2 \boldsymbol{E}) & \xrightarrow{\cong} & H^0(\Sigma, \boldsymbol{L} \otimes \boldsymbol{L}) \end{array}$$

Obviously, if $\boldsymbol{E}$ is holomorphically equivalent to $\boldsymbol{E}'$, then $T_{E,S} = T_{E',S'}$. *On the other hand, if* $\boldsymbol{T}_{E,S} = T_{E',S'}$ *and* $\boldsymbol{E}, \boldsymbol{E}'$ *are generated by their global sections, then* $\boldsymbol{E}$ is holomorphically equivalent to $\boldsymbol{E}'$. In fact, since $\boldsymbol{E}$ is generated by its global sections, there exists a mapping $\Sigma \to \boldsymbol{G}(2, \boldsymbol{p})$, *where* $\boldsymbol{G}(2, \boldsymbol{p})$ *is the Grassmann manifold of 2-dimensional subspaces of* C^p, *which can be embedded into* $\boldsymbol{P}(\wedge^2 \boldsymbol{C}^p)$. *This mapping induces an embedding of* Σ *into* $\boldsymbol{P}(\wedge^2 \boldsymbol{C}^p)$, *given by* $\boldsymbol{s}_i \wedge \boldsymbol{s}_j$. *Therefore* $\boldsymbol{T}_{E,S}$ *determines the isomorphic class of* $\boldsymbol{E}$.

Therefore, $M(2, 2\boldsymbol{l}, \boldsymbol{L} \otimes \boldsymbol{L})$ *can be embedded into* $P(Hom(\wedge^2 C^p, \boldsymbol{W}))$.

The action of $SL(\boldsymbol{p}, \boldsymbol{C})$ *on* C^p *induces actions on* $\wedge^2 \boldsymbol{C}^p$, $\mathbf{Hom}(\wedge^2 \boldsymbol{C}^p, \boldsymbol{W})$ *and* $\boldsymbol{P}(\mathbf{Hom}(\wedge^2 \boldsymbol{C}^p, \boldsymbol{W}))$.

Lemma 4　$\boldsymbol{E}$ is semi-stable (stable) if and only if $\boldsymbol{T}_{E,S}$ *is semi-stable* (*stable*) *with respect to the action of* $SL(p, \boldsymbol{C})$.

Proof　Here deal with the semi-stable case only. Let $\boldsymbol{E}$ be semi-stable, then $\boldsymbol{T}_{E,S}$ *is semi-stable. For* $\boldsymbol{E}$, there exists the Harder-Narasimhan filtration[6], that is, there exists a holomorphic line bundle $\boldsymbol{E}_1$ such that

$$0 \subset \boldsymbol{E}_1 \subset \boldsymbol{E}, \text{ and } \deg(\boldsymbol{E}_1) > \deg(\boldsymbol{E})/2$$

The assumption of $deg(\boldsymbol{E}) > 2(2g-1)$ implies $\deg(\boldsymbol{E}) > 2g-1$. By Lemma 3. $\dim H^1(\Sigma, \boldsymbol{E}_1) = 0$, and $\boldsymbol{E}_1$ *is generated by* $H^0(\Sigma, \boldsymbol{E}_1)$. *From the fact that* $\boldsymbol{H}^0(\Sigma, \boldsymbol{E}_1) \subset H^0(\Sigma, \boldsymbol{E})$ *and the Riemann-Roch Theorem*,

$$\dim H^0(\Sigma, E_1) = \deg(E_1) + (1-g) > \deg(E)/2 + (1-g) = (\deg(E) + 2(1-g))/2 = p/2$$

By paper [6], it's obvious that the above inequality contradicts the assumption that $T_{E,S}$ *is semi-stable.*

Lemma 5 The moduli space of semi-stable holomorphic vector bundles of rank 2, degree $2l$ and the determinant bundle $L \otimes L$ over Σ under the S-equivalence can be embedded into the space $P(\mathbf{Hom}(\wedge^2 C^p, W))//SL(p, C)$.

Proof First, if E is stable, from Lemma 1.9 of paper [3], $T_{E,S}$ *is stable, hence the set* $\pi^{-1}(\pi(T_{E,S}))$ *consists exactly of the orbit of* $T_{E,S}$ *in* $P(\mathbf{Hom}(\wedge^2 C^p, W))$, *where*

$$\pi: P(\mathbf{Hom}(\wedge^2 C^p, (W)) \to P(\mathrm{Hom}(\wedge^2 C^p, W))//SL(p, C)$$

$\pi(T_{E,S})$ *corresponds to the isomorphic class of* E. If E is semi-stable, so is $T_{E,S}$. *In this case* $\pi^{-1}(\pi(T_{E,S}))$ *consists of some orbits in* $P(\mathbf{Hom}(\wedge^2 C^p, W))$. *If* O, Q are semi-stable points in $P(\mathbf{Hom}(\wedge^2 C^p, W))$, then $\pi(O) = \pi(Q)$ *if and only if the closure of the SL*(p, C)-orbit of O intersects the closure of the SL(p, C)-orbit of Q.

According to paper[4], Let $O = T_{E,S}$, $Q = T_{E',S'}$, *then the closure of the SL*(p, C)-orbit of O intersects the closure of the SL(p, C)-orbit of Q if and only if $\mathrm{gr}E = \mathrm{gr}E'$. This completes the proof.

The set of the conjugate classes of irreducible represention s is an open $(3g-3)$*-dimensional algebraic manifold*[2], *therefore its compactification* M_0 *is also a projective variety.*

References:

[1] DONADSON S K. The geometry of four-manifolds[M]. London: Oxford University Press, 1990. 1 ~ 99.

[2] NARASIMHAN M S, SESHADRI C S. Stable and unitary vector bundles on a compact surface[J]. Ann Math, 1965, 82: 540 ~ 567.

[3] SESHADRI C S. Space of unitary vector bundles on a compact Riemann surface[J]. Ann Math, 1967, 84: 303 ~ 336.

[4] MUMFORD D. Geometric invariant theory[M]. New York: Springer Verlag, 1965. 1 ~ 217.

[5] GIESEKER D. Geometric invariant theory and applications to moduli problems[M]. New York: Springer-Verlag, 1982. 45 ~ 73.

[6] HARDER G, NARASIMHAN M S. On the cohomology groups of moduli spaces of vector bundles over curves[J]. Math Ann, 1975, 212: 215 ~ 248.

表示空间与全纯向量丛

胡新民*

摘 要: 证明了完备、定向曲面的基本群在SU(2) 中的表示空间可等同于该曲面上秩为2且具有平凡行列式丛的全纯向量丛的模空间. 并且它们具有射影结构.

关键词: 模空间; 酉表示; (半) 稳定
中图分类号: O189. 3; O152. 6

* 中山大学数学系, 广州 510275